View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 306 (2006) 2786–2789
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

ρ-Valuations for some stunted trees

André E. Kézdy

Department of Mathematics, University of Louisville, Louisville, KY 40292, USA

Received 22 December 2005; received in revised form 22 May 2006; accepted 24 May 2006
Available online 24 July 2006

Abstract

A tree with n edges is *stunted* if its edges can be linearly ordered $e_1, \ldots, e_n$ so that e_1 and e_2 share a vertex and, for all $j = 3, \ldots, n$, edge e_j shares a vertex with at least one edge e_k satisfying $2k \leqslant j - 1$. Using Alon's "Combinatorial Nullstellensatz", a short proof is given showing that if $p = 2n + 1$ is prime, then every stunted tree with n edges has a ρ-valuation. Consequently, every stunted tree on n edges cyclically decomposes the complete graph K_p.
© 2006 Elsevier B.V. All rights reserved.

Keywords: Tree labeling; Combinatorial nullstellensatz; Graph decompositions

1. Introduction

A ρ-valuation of a graph G with n edges is an injection of the vertex set into the set $\{0, \ldots, 2n\}$ so that, if the edge labels induced by the absolute value of the difference of the vertex labels are $\ell_1, \ldots, \ell_n$, then $\ell_i = i$ or $\ell_i = 2n + 1 - i$. Rosa [6] introduced this notion as one of several in a hierarchy of successively weaker valuations of graphs, $\alpha-$, $\beta-$, $\sigma-$, and $\rho-$. The first three of these valuations are not defined here (the interested reader can refer to Gallian's survey [2]). Note that β-valuations are often referred to as *graceful labelings*.

While some trees do not have α-valuations (see [3]), the famous Graceful Tree Conjecture states that all trees have β-valuations. However, this conjecture is still open and has resisted much serious effort, which suggests that perhaps more effort should be devoted to the search for counterexamples. Some modest families of trees have been shown to have β-valuations: caterpillars, symmetrical trees, small fixed diameter trees, trees with a small fixed number of leaves, trees with at most 27 vertices, and a host of similar results, too numerous to mention here, which have been cataloged in Gallian's dynamic survey [2]. Because these trees have β-valuations, they have ρ-valuations, but apparently not much more is known about trees with ρ-valuations. In particular, it is not known whether all trees have ρ-valuations; indeed this is a conjecture usually credited to Kotzig [4].

A *decomposition* of a graph $G = (V, E)$ is a partition of E into pairwise edge disjoint subgraphs. If these edge disjoint subgraphs are all isomorphic to the same graph H, then we say that H *decomposes* G. One of the most famous conjectures about decomposing graphs is Ringel's conjecture [5] which states that every tree on n edges decomposes the complete graph on $2n+1$ vertices, K_{2n+1}. Ringel's conjecture remains open and has been the motivation behind many of the conjectures on labeling trees. A tree T on n edges *cyclically decomposes* K_{2n+1} if there exists an

E-mail address: kezdy@louisville.edu.

0012-365X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2006.05.026

injection $\psi : V(T) \to \mathbb{Z}_{2n+1}$ such that, for all distinct $i, j \in \mathbb{Z}_{2n+1}$, there exists a unique $k \in \mathbb{Z}_{2n+1}$ with the property that there is a pair of adjacent vertices $u, v \in V(T)$ satisfying $\{i, j\} \equiv \{\psi(u) + k, \psi(v) + k\}$. Rosa [6] proved that a tree on n edges cyclically decomposes K_{2n+1} if and only if it has a ρ-valuation.

A tree with n edges is *stunted* if its edges can be linearly ordered $e_1, \ldots, e_n$ so that e_1 and e_2 share a vertex and, for all $j = 3, \ldots, n$, edge e_j shares a vertex with at least one edge e_k satisfying $2k \leqslant j - 1$. While this definition gives an easy algorithm to construct many stunted trees, the family of stunted trees is rather small (a stunted tree with n edges has diameter at most $2\log_2(n)$, whereas the random tree has diameter of order $\sqrt{n}$), but rich in the sense that they may be asymmetrical, may have a complicated base (the base of a tree is the tree that remains after removing all leaves), may have an unbounded diameter, and may have an unbounded number of leaves. These rich properties together with the technique used to prove they have ρ-valuations bring stunted trees into sharp contrast with other families of trees known to have ρ-valuations.

Using Alon's "Combinatorial Nullstellensatz", a short proof is given in the next section that shows that if $p = 2n + 1$ is prime, then every stunted tree with n edges has a ρ-valuation. Consequently, every stunted tree on n edges cyclically decomposes the complete graph K_p.

2. Stunted trees

In this section we make use of the following result.

Theorem 1 (*Alon [1]*). *Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer. If the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero, then for any subsets $S_1, S_2, \ldots, S_n$ of F satisfying $|S_i| > t_i$, there are elements $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ such that*

$$f(s_1, s_2, \ldots, s_n) \neq 0.$$

It is useful to give an alternate (but equivalent) definition of a ρ-valuation: a ρ-valuation of a graph G on n edges is an injection $\rho : V(G) \to \{0, \ldots, 2n\}$ so that the induced edge labels $\rho_e = \rho(u) - \rho(v)$, for $uv = e \in E(G)$, satisfy

$$\rho_e \not\equiv \pm\rho_f \pmod{2n + 1},$$

for all distinct $e, f \in E(G)$.

Theorem 2. *If $2n + 1$ is a prime integer and T is a stunted tree with n edges, then T has a ρ-valuation.*

Proof. Let T be a stunted tree on n edges, where $p = 2n + 1$ is prime. By definition, the edges $e_1, e_2, \ldots, e_n$ of T can be indexed so that e_1 and e_2 share a vertex and, for all $j = 3, \ldots, n$, edge e_j shares a vertex with at least one edge e_k satisfying $2k \leqslant j - 1$. This ordering of the edges means that, for all $i = 1, \ldots, n$, the graph induced by the edges $e_1, e_2, \ldots, e_i$ is a subtree of T; it is denoted $T[i]$. Moreover, because T is stunted, for any $i = 1, \ldots, n$ and any index j satisfying $i < j \leqslant 2i + 1$, the graph $T[i] + e_j$ (obtained by adding the edge e_j to the tree $T[i]$) is a subtree of T.

Now e_1 and e_2 share a vertex, v_0, which we view as the root of the tree T. By definition, the subtree $T[i]$ contains the root, for all $i = 1, \ldots, n$. Order the remaining vertices this way: the vertex v_1 is the nonroot vertex incident to e_1, the vertex v_2 is the nonroot vertex incident to e_2, and, for $i = 3, \ldots, n$, the vertex v_i is the vertex in $V(T[i]) \setminus V(T[i - 1])$. So the vertex set of T is $\{v_0, \ldots, v_n\}$. For any vertex v_i, define the set

$$P(i) = \{j: \text{ edge } e_j \text{ appears on the path of } T \text{ connecting } v_i \text{ with } v_0\};$$

so $P(i)$ is the set of indices for edges on the unique v_0v_i-path in T. Notice that $P(0) = \emptyset$, by definition.

To each edge e_j assign a variable x_j. For each vertex v_i define the polynomial $g_i \in \mathbb{Z}_p[x_1, \ldots, x_n]$ this way:

$$g_i(x_1, \ldots, x_n) = \sum_{j \in P(i)} x_j.$$

Since $P(0) = \emptyset$, we define $g_0(x_1, \ldots, x_n) = 0$. We use g_i as an abbreviation for $g_i(x_1, \ldots, x_n)$.

Now define the polynomial

$$f_T(x_1,\ldots,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_j^2-x_i^2)\prod_{0\leqslant i<j\leqslant n}(g_j-g_i).$$

The motivation for this definition of f_T is the claim that T has a ρ-valuation if and only if $f_T(a_1,\ldots,a_n)\not\equiv 0 \pmod p$, for some $(a_1,\ldots,a_n)\in\mathbb{Z}_p^n$. To see this, observe that, if $f_T(a_1,\ldots,a_n)\not\equiv 0 \pmod p$, then one can define the mapping $\rho: V(T)\to\mathbb{Z}_p$ by setting $\rho(v_i)=g_i(a_1,\ldots,a_n)$. The factor $\prod_{0\leqslant i<j\leqslant n}(g_j-g_i)$ of f_T guarantees that ρ is an injection. The induced edge labels are $a_1,a_2,\ldots,a_n$, so they satisfy

$$a_i\not\equiv\pm a_j \pmod p,$$

for all $1\leqslant i<j\leqslant n$ because $\prod_{1\leqslant i<j\leqslant n}(a_j^2-a_i^2)\not\equiv 0 \pmod p$. Similarly, the existence of a ρ-valuation determines induced edges labels $a_1,a_2,\ldots,a_n$ that guarantee $f_T(a_1,\ldots,a_n)\not\equiv 0 \pmod p$. Thus, it suffices to show that $f_T\not\equiv 0$.

Actually, we prove that the related polynomial

$$F_T(x_1,\ldots,x_n)=f_T(x_1,\ldots,x_n)\left(\prod_{i=1}^n x_i^i\right)$$

does not vanish on all inputs from $\mathbb{Z}_p$; this implies $f_T\not\equiv 0$. (Note: the extra factor, $\prod_{i=1}^n x_i^i$, added to F_T is not necessary, and a stronger theorem could be proven by omitting it, but this direction is not pursued for the sake of simplicity.) The total degree of F_T is $2n^2$. We shall prove that the absolute value of the coefficient of the monomial $\prod_{i=1}^n x_i^{2n}$ is 1. It will follow from Theorem 1 with $S_i=\mathbb{Z}_p$, for all $i=1,\ldots,n$, that $F_T\not\equiv 0$.

First observe that, for $0\leqslant i<j\leqslant n$, the factor (g_j-g_i) is simply a linear factor in which variable x_k appears with nonzero coefficient if and only if edge e_k is on the unique path of T connecting vertex v_i with vertex v_j. Define for $0\leqslant i<j\leqslant n$,

$$Q_{ij}=(g_j-g_i)=\sum_{k\in P(j)}x_k-\sum_{k\in P(i)}x_k.$$

For convenience, we partition the factors of F_T into products labeled V, Q, and R:

$$\underbrace{\left(\prod_{1\leqslant i<j\leqslant n}(x_j^2-x_i^2)\right)}_{V}\underbrace{\left(\prod_{0\leqslant i<j\leqslant n}Q_{ij}\right)}_{Q}\underbrace{\left(\prod_{i=1}^n x_i^i\right)}_{R}.$$

It follows from Vandermonde's identity that

$$V=\sum_{\pi\in\mathscr{S}_n}\operatorname{sign}(\pi)\prod_{k=1}^n x_{\pi(k)}^{2(n-k)},$$

where $\mathscr{S}_n$ denotes the set of permutations of $\{1,\ldots,n\}$. Thus, to produce the monomial $\prod_{i=1}^n x_i^{2n}$ in the expansion of F_T, a monomial $\prod_{k=1}^n x_{\pi(k)}^{2(n-k)}$ in V must match up with a monomial in the expansion of QR; so the latter must have the form $\prod_{k=1}^n x_{\sigma(k)}^{2k}$, for some $\sigma\in\mathscr{S}_n$. Consider such a monomial in the expansion of the product QR. We shall prove that it is unique, namely that it must be $\prod_{k=1}^n x_k^{2k}$. This will follow from the following claim.

Claim. *If, for some $\sigma\in\mathscr{S}_n$, the monomial $M=\prod_{k=1}^n x_{\sigma(k)}^{2k}$ occurs in the expansion of QR, then for all $k=1,\ldots,n$, the monomial M is obtained from the expansion of QR by selecting x_k in those factors of Q corresponding to v_iv_k-paths $(i=0,\ldots,k-1)$ in T, and no other factors.*

The proof of the claim is by induction on k.

Basis step: $k=1$. Because $x_{\sigma(1)}$ appears only to the second power in M and the factor R has x_j to the jth power, it follows that $\sigma(1)=1$ or $\sigma(1)=2$. But $\sigma(1)\neq 2$ since x_2 appears to the second power in R and also as a factor of Q

corresponding to the path in T connecting the endpoints of e_2. Thus, $\sigma(1)=1$ and, to form M, the variable x_1 can only be chosen in the factor of Q corresponding to the one v_0v_1-path in T.

Induction step: $k>1$. Because $x_{\sigma(k)}$ appears to the $2k$th power in M and the factor R has too many occurrences of x_j, for $j>2k$, it follows that $1\leqslant\sigma(k)\leqslant 2k$. Now the induction hypothesis implies that $\sigma(j)=j$, for all $j=1,\dots,k-1$ and further, to form M, the variable x_j must be chosen from those factors of Q corresponding to v_iv_j-paths in T, $i=0,\dots,j-1$, and no other factors. Consequently, $k\leqslant\sigma(k)\leqslant 2k$.

Suppose then that $\sigma(k)=j$, for some j satisfying $k\leqslant j\leqslant 2k$. Because T is stunted, the graph $T[k-1]+e_j$ is a subtree of T; it has $k+1$ vertices, namely $v_0,\dots,v_{k-1},v_j$. Now x_j appears to the jth power in R. Furthermore, for each path connecting v_i to v_j in $T[k-1]+e_j$ (for $i=0,\dots,k-1$), there is a corresponding linear factor of Q containing x_j from which x_j must be selected to form M because all of the other variables cannot be selected again, as the inductive hypothesis guarantees. Thus, x_j must appear in QR to the power at least as large as $j+k$. This forces $\sigma(k)=k$ and the claim is proven. □

Corollary 1. *Suppose $2n+1$ is a prime integer and T is a stunted tree on n edges ordered $e_1,\dots,e_n$. If $S_1,\dots,S_n$ are subsets of $\{0,\dots,2n\}$ such that $|S_i|=i$, then T has a ρ-valuation in which edge e_i avoids labels in S_i.*

Proof. Apply the argument in the proof above with the polynomial f_T but a new factor R given by

$$R=\prod_{i=1}^{n}\prod_{s\in S_i}(x_i-s). \qquad \square$$

Kotzig [4] proved that, for any tree T and any edge e of T, every sufficiently long subdivision of e yields a tree with an α-valuation (hence a ρ-valuation). Here we prove a weaker complementary result: adding sufficiently many leaves to any fixed vertex of a tree eventually leads to a tree with a ρ-valuation.

Corollary 2. *Let T be a finite tree and v an arbitrary vertex of T. There exists a constant K depending on T and v such that, if $2n+1$ is prime and T' is a tree with n edges that is obtained from T by adding at least K pendent leaves at v, then T' has a ρ-valuation.*

Proof. Simply observe that there exists a constant K such that, if T' is a tree obtained from T by adding at least K pendent leaves at v, then T' is stunted. □

Acknowledgment

The author would like to thank Hunter Snevily for the helpful discussions.

References

[1] N. Alon, Combinatorial nullstellensatz, Combin. Probab. Comput. 8 (1–2) (1999) 7–29.
[2] J. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 5 (2005) Dynamic Survey 6.
[3] C. Huang, A. Kotzig, A. Rosa, Further results on tree labelings, Utilitas Math. 21 (1982) 31–48.
[4] A. Kotzig, On certain vertex-valuations of finite graphs, Utilitas Math. 4 (1973) 261–290.
[5] G. Ringel, Problem 25, in: Theory Graphs and Its Applications, Proceedings of Symposium Smolence 1963, Czech. Acad. Sci. (1964) 162.
[6] A. Rosa, On certain valuations of the vertices of a graph, Theory Graphs (International Symposium, Rome, 1966) (1967) 349–355.